The Wilsonian Flux

Issue 1 / December 2024

Welcome to the first issue of The Wilsonian Flux! The publication will aim to spark an interest and foster curiosity in physics and engineering beyond what's taught at school. Physics and relevant maths will be covered, introducing you to more complex ideas. As well as this, there will be problems to solve at the end of every issue. I hope you enjoy!

Each issue will contain at least one in-depth physical topic explanation and an explanation of a more advanced mathematical method. At some points, they may feel like fleetingly quick explanations - this is because I don't want this to read too much like a textbook. If something genuinely interests you, I encourage you to research and learn more about the topic yourself.

Contents

Figure 1: Pauli and Bohr playing with a spinning top in May 1951, at ages 51 and 65 [1]

1 Exploring rotational motion

Here, I explore some concepts surrounding rotational mechanics and dynamics to a greater level of depth than A-Level, hopefully developing your intuition of rotational systems further.

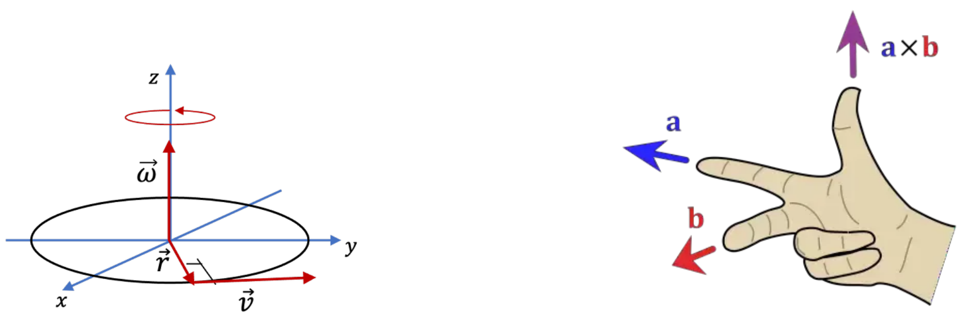
1.1 Axial Vectors

Vectors representing rotational quantities are different from the conceptual understanding of vectors that is developed throughout school. Yes, angular velocity and acceleration both have a direction and magnitude - they are vectors - but, the direction of these vectors are not the same as the direction of their tangential counterparts as you might originally assume. Instead, the direction of these rotational vectors is actually along the axis of rotation. Consequently, vectors of this kind are called axial vectors.

To see why this is case, we need to formalise our definition of angular velocity and acceleration as vectors, ensuring that our definitions of them do indeed yield a vector. The A-Level definitions of these quantities operate only in terms of magnitudes: $\omega = \frac{v}{r}$, gives only the magnitude of the angular velocity ω, given the magnitudes of the tangential velocity v and the distance from axis of rotation r. This logic follows for angular acceleration. Instead we shall define angular velocity and angular acceleration as follows:

$$\vec{\omega} = \frac{1}{|\vec{r}|^2}\vec{r} \times \vec{v}$$
$$\vec{\alpha} = \frac{1}{|\vec{r}|^2}\vec{r} \times \vec{a}$$

We define the vector $\vec{r}$ for a particle to originate from the axis of rotation and end at the particle - it is perpendicular to the axis of rotation. $\vec{v}$ and $\vec{a}$ are the tangential velocity and acceleration. As we can see, the angular velocity is defined as the cross product between $\vec{r}$ and $\vec{v}$. To find the direction of a vector cross product, we need only the direction of the vectors and our right hand. For a vector cross product $\vec{c} = \vec{a} \times \vec{b}$, using your right hand, point your index finger in the direction of $\vec{a}$ and middle finger in the direction of $\vec{b}$. The direction your thumb points is the direction of the cross-product $\vec{c}$. Applying this to our definitions of angular velocity and acceleration, you can observe that your thumb will point along the axis of rotation, making them axial vectors.



(a) Vector diagram of a rotational system [2]

(b) Right hand rule for $\vec{a} \times \vec{b}$ [3]

Figure 2: Axial vectors

It is important to note the importance of the direction of axial vectors. When viewed from above, rotation occurs anticlockwise around an axial vector. Consequently, convince yourself the cross product is anti-commutative. I will also leave you to develop an intuition for where the $\frac{1}{|\vec{r}|^2}$ term comes from in our definitions. (Hint: $\vec{v} = \vec{\omega} \times \vec{r}$).

1.2 Torque: rotational force

Torque is also something that is studied at A-Level, but once again, convinced it is a vector, we need to establish its direction relative to the rotation. Unsurprisingly perhaps, torque is also an axial vector! Torque also lends itself to a definition as the cross product of two vectors: the position and the force.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Through this definition, we see that torque follows the convention for axial vectors i.e. a torque vector pointing in the positive z direction will correspond to a force being applied anticlockwise relative to the origin in the xy-plane. It may be useful at this point to state another property of the vector cross product, namely, for $\vec{a} \times \vec{b}$, the following inequality holds:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

θ is the angle between the two vectors. Using this, it becomes possible to derive the equations involving magnitudes, which we learn, such as $\tau = Fd\sin\theta$.

Torque as a concept is the rotational analogue of force in a lot of ways, as we shall see later. Just like a force causes linear motion through space, torque causes rotational motion around an axis. Just like the force applied to an object determines its acceleration, the torque applied to an object determines its angular acceleration.

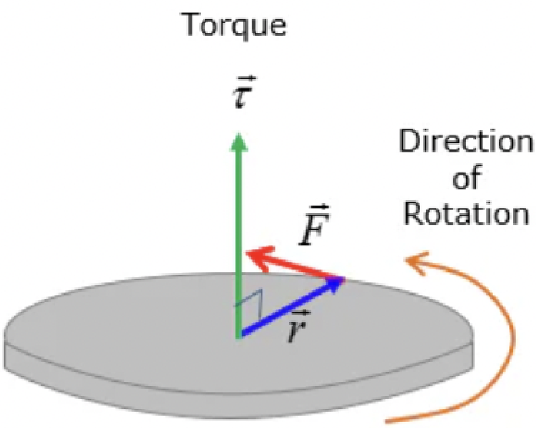


Figure 3: Torque as an axial vector [4]

1.3 Moment of inertia: rotational mass

This may potentially be an entirely new concept to some of you. Similar to how inertial mass quantifies a body's "resistance" to a force causing it to accelerate, the moment of inertia quantifies how a mass is distributed about an axis of rotation, and consequently tells us how "hard" it is for a torque to cause it to rotationally accelerate.

For a single point mass m a distance r from the axis of rotation, it's moment of inertia is defined as:

$$I = mr^2$$

For example, consider a particle of mass 5kg attached to a rod of negligible mass of length 1m. If we let the rod rotate about its midpoint, the moment of inertia of this particle would be $5\text{kg} \cdot (0.5\text{m})^2 = 1.25\text{kg m}^2$. We note that moment of inertia has the dimensions $M^1L^2T^0$

(this means it has units kg m^2). We can extend this to a system of multiple discrete point masses, where each would contribute separately to the moment of inertia of the system about an axis: For a system of n masses:

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Unfortunately, not every system that rotates is made from a combination of point masses. Let us consider how to calculate the moment of inertia of a continuous body about an axis of rotation. Imagining we split a continuous body into an infinite number of particles, each with mass dm, the moment of inertia logically follows:

$$I = \int r^2 \, \mathrm{d}m$$

Integrating over all of the mass elements acts as a way of carrying out an infinite, weighted sum of each mass element relative to its distance from the axis squared. This is best demonstrated with an example: we shall calculate the moment of inertia of a uniform rod of mass M and length L rotating around its center.

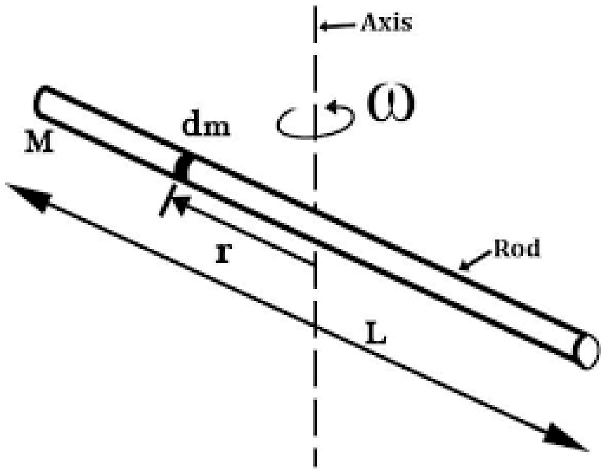


Figure 4: A uniform rod of mass M and length L [5]

First, divide the rod into small elements, taking advantage of the fact that its density is uniform:

$$\mathrm{d}m = \frac{M}{L} \, \mathrm{d}r$$

Then setup our integral and solve:

$$\begin{aligned} I &= \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, \mathrm{d}m \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \left(\frac{M}{L}\right) \, \mathrm{d}r \\ &= \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \, \mathrm{d}r \\ &= \frac{M}{L} \left[\frac{1}{3} r^3\right]_{r=-\frac{L}{2}}^{\frac{L}{2}} \\ &= \frac{M}{L} \left[\frac{L^3}{12}\right] = \boxed{\frac{1}{12} M L^2} \end{aligned}$$

Therefore the moment of inertia for any uniform rod rotating about its centre with mass M and length L is $\frac{1}{12}ML^2$. Remember, the moment of inertia depends on where the axis of rotation is!

Here are the rotational inertia for different solid bodies - hopefully this will help you appreciate how the shape of the body can dramatically affect how hard it is to rotate.

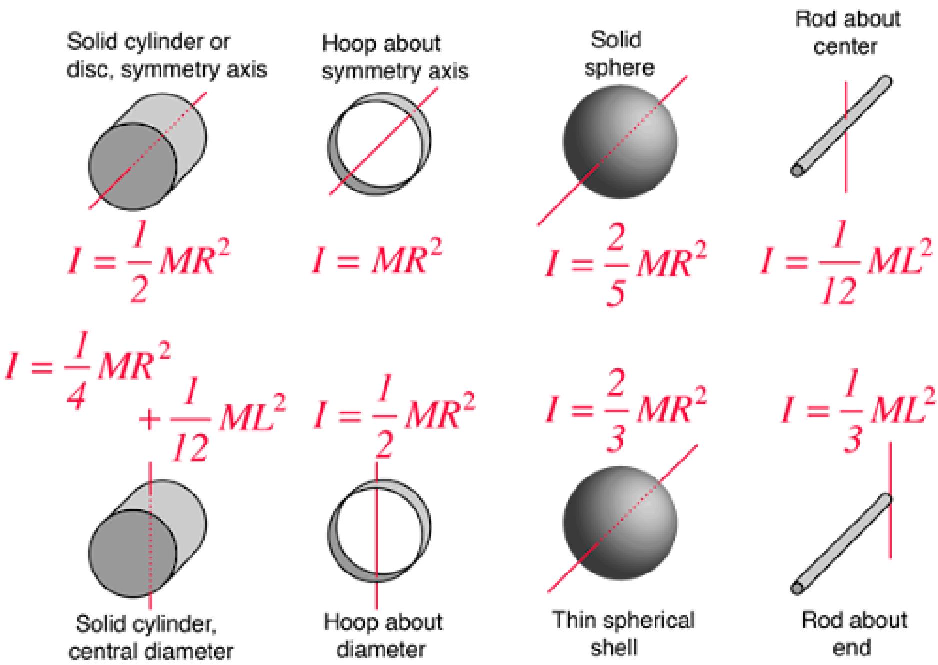


Figure 5: Moment of inertia of various solid bodies [6]

1.4 Rotational Newton II

We have now built up a substantial amount of theoretical knowledge regarding both torque and moment of inertia. The rotational quantity that links both of these concepts together is angular acceleration.

$$\vec{\tau} = I\vec{\alpha}$$

Perhaps you can see the resemblance of this equation to the constant-mass form of Newton's second Law: $\vec{F} = m\vec{a}$. They can be considered analogous: just like it would take 1N of force to accelerate a mass of 1kg at 1ms^{-2}, it would take 1Nm of torque to accelerate a body with moment of inertia 1 kg m^2 at 1 rad s^{-2}.

Similar to Newton's First Law, we can obtain a rotational analogue, that is: if the sum of the torques acting around an axis is 0, there will be no resultant angular acceleration - this is synonymous to the principle of moments.

Unfortunately, the above equation for torque assumes that the moment of inertia is constant throughout the motion. Sometimes, this is not the case: for example, when you spin on a spinning chair and then pull your hands in, you change your moment of inertia. To remove our assumption that moment of inertia is constant, we will have to explore the concept of angular momentum.

1.5 Angular momentum

Angular momentum is an axial vector, and acts as the rotational equivalent of linear momentum. Linear momentum $\vec{p} = m\vec{v}$, but the definition of angular momentum is:

$$\vec{L} = I\vec{\omega}$$

However, angular momentum can also be defined as the vector product:

$$\vec{L} = \vec{r} \times \vec{p}$$

Just as linear momentum is the tendency for a body to continue moving, the angular momentum simply quantifies the tendency of a body to continue rotating. It is important to notice the difference in dimensions of angular and linear momentum: linear momentum has dimensions $M^1L^1T^{-2}$ and angular momentum has dimensions $M^1L^2T^{-1}$. This can be derived from the formulae above.

We can link angular momentum to torque, similar to how Newton's 2nd Law links linear momentum to force:

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

Torque causes a change in angular momentum, just like force causes a change in linear momentum, and consequently for a system where $\vec{\tau} = 0$, the change in $\vec{L}$ is also 0: angular momentum is conserved.

The conservation of angular momentum is best illustrated through an example. An ice skater rotating with their arms spread out pulls their arms in - what happens? The ice skater has decreased their moment of inertia by distributing their mass closer to the axis of rotation, and in order to conserve angular momentum, their angular velocity will increase i.e. they will spin faster. This effect can also be seen on cosmic scales: as a star dies and it collapses, its moment of inertia decreases, but in order to conserve angular momentum, its angular velocity increases - this can sometimes lead to the formation of rapidly rotating neutron stars.

There's a lot more to spinning

Admittedly, we have barely scratched the surface of the physics behind rotational motion, but there is no shortage of concepts to uncover: rotational kinetic energy, gyroscopic motion like precession and nutation, the rotating reference frame and pseudoforces (the centrifugal and coriolis force) and even quantum rotational motion, where quantities such as angular momentum are quantised.

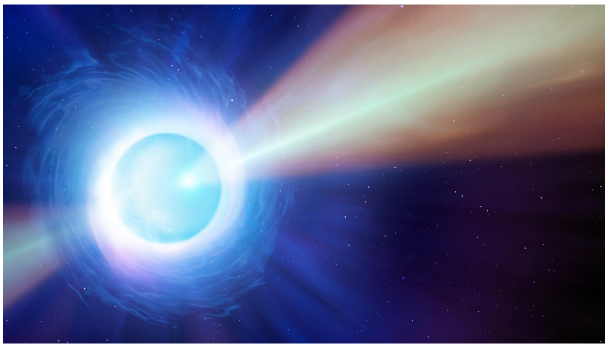


Figure 6: A rapidly rotating neutron star, known as a pulsar [7]

2 Co-ordinate systems

For most of your education, you will have used Cartesian co-ordinates, where a point in 2-dimensional space can be defined by an x coordinate and a y coordinate. We can add an additional z-axis for 3-dimensional space. I will assume you are familiar and comfortable with the Cartesian co-ordinate system in order to explore further co-ordinate systems: polar, spherical and cylindrical. The choice of using a different co-ordinate system relies entirely on the problem at hand - if a problem can be solved more easily by using a different co-ordinate system, it might be sensible to use it.

2.1 Polar

Polar co-ordinates are a way of defining a point in 2D space using its radial distance from the origin (the pole), r and its angular displacement from (conventionally) the positive x-axis (the polar axis) going anti-clockwise, θ. A lot of rotational motion is much better suited to the use of polar co-ordinates - you may even be doing this without realising it.

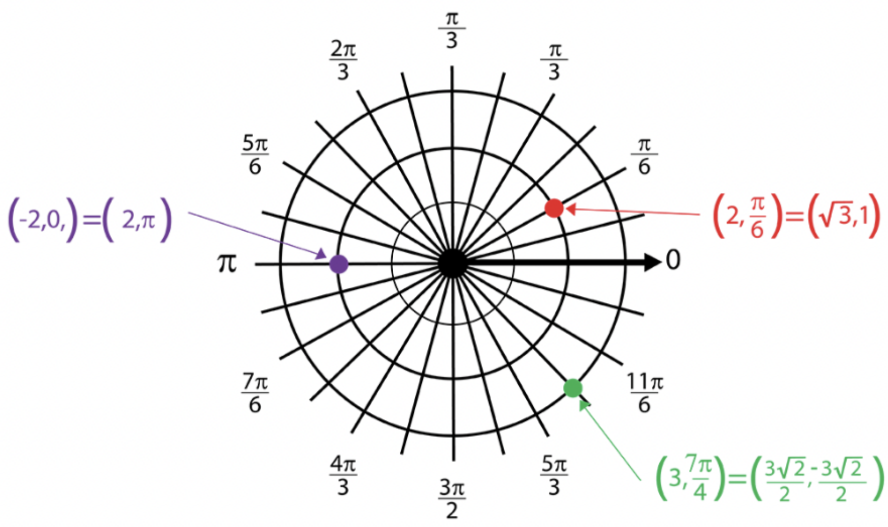


Figure 7: Cartesian and polar representation of coordinates: $(r, \theta) = (x, y)$ [8]

Figure 7 shows the polar grid system containing radial lines, where each represents a different θ and concentric circles centred on the origin, each representing a different r. The transformations from Cartesian (x, y) to polar co-ordinates (r, θ) are $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan \frac{y}{x}$.

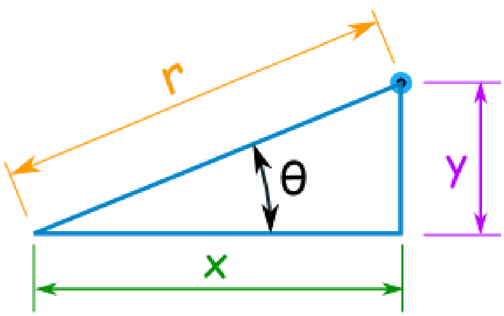
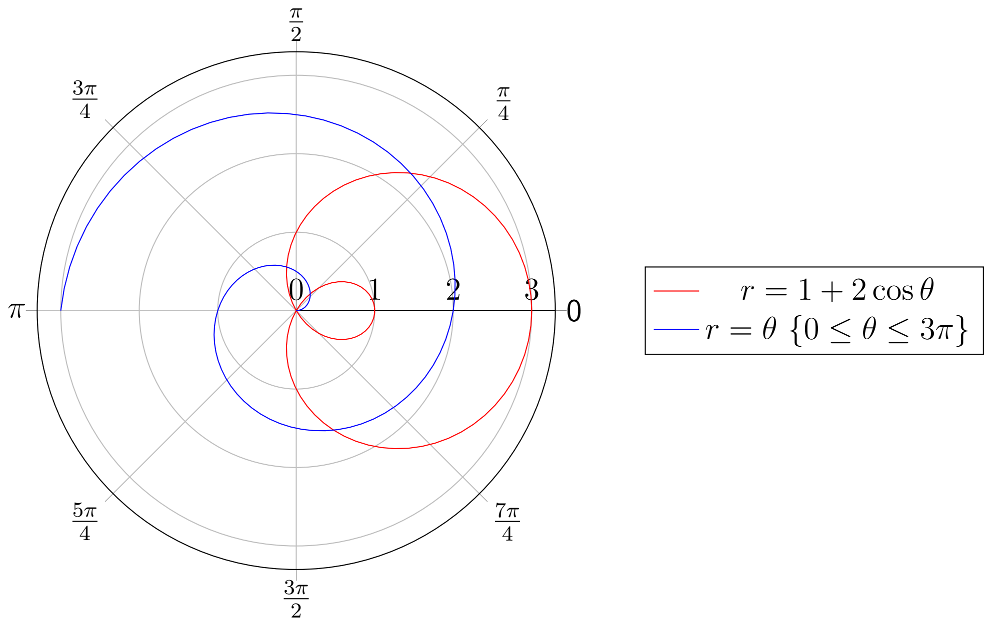


Figure 8: A point with its Cartesian and polar co-ordinates labelled [9]

Observe that the points $(1, \pi)$, $(1, 3\pi)$ and $(1, -\pi)$ are equivalent in the polar co-ordinate system. By convention, when defining points, we keep $0 \leq \theta \leq 2\pi$. Similar to how a curve in Cartesian co-ordinates provides a relationship between the x and the y co-ordinates, curves defined in polar co-ordinates provide a relationship between θ and r.

Here are a few examples of polar curves and their corresponding equations. I'd encourage you to visit Desmos and play around with these - it's possible to generate some really interesting curves (what does $r = \sin 1.8^\theta - 1.8$ look like?)



The transformations for converting back to Cartesian co-ordinates from polar co-ordinates follow on logically from earlier, and are basic trigonometry. They are:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

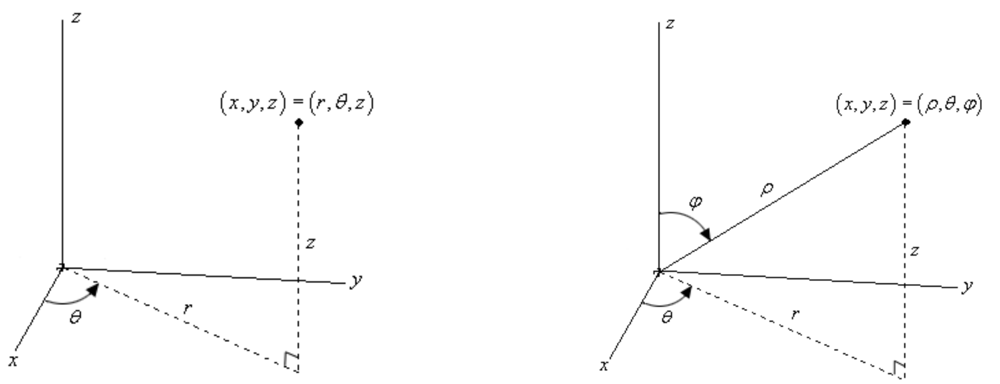
Polar co-ordinates can be used to simplify problems involving circular motion or almost-circular (like a spiral or cardioid) motion.

2.2 Spherical and Cylindrical

The spherical and cylindrical co-ordinate systems both act as 3D extensions of the polar co-ordinates in different ways.

Cylindrical co-ordinates

Cylindrical co-ordinates extend the polar co-ordinate system to 3D through the addition of an extra z axis, such that a point in space is now described as (r, θ, z). z is simply the vertical distance of the plane from the xy-plane i.e. the z-axis acts exactly as it does in a 3D Cartesian co-ordinate system. As such, converting from Cartesian to cylindrical is identical to converting from Cartesian to polar, just with the additional mapping of $z = z$. Cylindrical co-ordinates are useful when describing helical motion, such as the motion of a charged particle in a solenoid, where the magnetic field induces a helical path.



(a) The cylindrical co-ordinate system [10] (b) The spherical co-ordinate system [11]

Figure 9: Alternate co-ordinate systems

Spherical co-ordinates

The cylindrical co-ordinate system added an additional z-axis to polar co-ordinates, whereas spherical co-ordinates instead include another angle, ϕ, referred to as the azimuthal angle. This angle is measured from the positive z-axis. Another change that occurs is while we used r in polar co-ordinates as a measure of distance from the origin, we will now use ρ, which takes the value of $\rho = \sqrt{x^2 + y^2 + z^2}$ as opposed to $r = \sqrt{x^2 + y^2}$.

Transforming co-ordinates from Cartesian to spherical and back requires some trigonometry, which I won't write out in full. The final results are below, but I encourage you to derive them yourself.

$$
\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2} & x &= \rho \sin\phi \cos\theta \\
\theta &= \arctan\frac{y}{x} & y &= \rho \sin\phi \sin\theta \\
\phi &= \arccos\frac{z}{\rho} & z &= \rho \cos\phi
\end{aligned}
$$

2.3 Application

Here, I'll give a worked example showing how alternate co-ordinate systems can sometimes be useful.

Question

Calculate the volume of a sphere of radius R using integration.

Answer

Using spherical co-ordinates, we can consider an infinitesimally small volume element of the sphere $\mathrm{d}V$. We know that three variables will affect the volume of $\mathrm{d}V$: these are $\mathrm{d}\rho$, $\mathrm{d}\theta$ and $\mathrm{d}\phi$.

Using some trigonometry and the arc length formula $s = r\theta$, it is possible to calculate the volume of $\mathrm{d}V$:

$$\mathrm{d}V = (\mathrm{d}\rho)(\rho\ \mathrm{d}\phi)(\rho \sin\phi) = \rho^2 \sin\theta\ \mathrm{d}\rho\ \mathrm{d}\theta\ \mathrm{d}\phi$$

Now see that the volume of the whole sphere can be made by integrating the small volume element from $\rho = 0$ to $\rho = R$, then by integrating from $\theta = 0$ to $\theta = \pi$ (at this point we

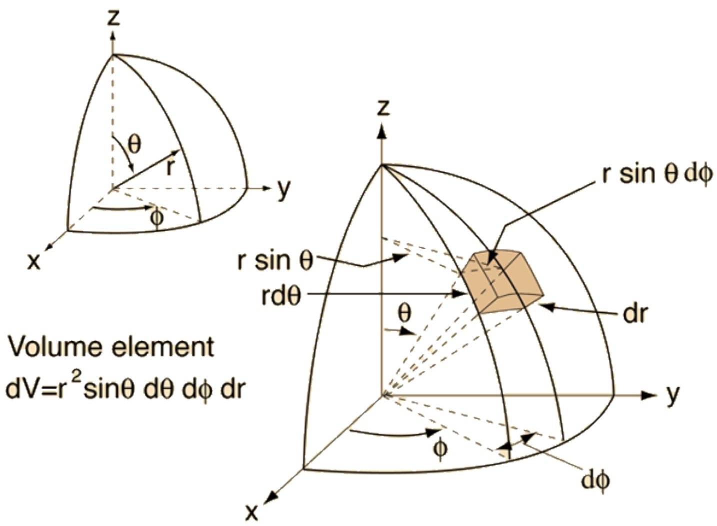


Figure 10: An infinitesimally small volume element $\mathrm{d}V$ of a sphere. [12]

have a semi-circle). Finally we revolve this shape around the y-axis by integrating from $\phi = 0$ to $\phi = 2\pi$.

$$
\begin{aligned}
V &= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\theta \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\frac{1}{3}\rho^3 \sin\theta\right]_{\rho=0}^{\rho=R} \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left[\frac{1}{3}\rho^3 \sin\theta\right]_{\rho=0}^{\rho=R} \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{1}{3}R^3 \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&= \frac{1}{3}R^3 \int_{\phi=0}^{\phi=2\pi} \left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} \ \mathrm{d}\phi \\
&= \frac{1}{3}R^3 \int_{\phi=0}^{\phi=2\pi} 2 \ \mathrm{d}\phi \\
&= \frac{1}{3}R^3 \left[2\phi\right]_{\phi=0}^{\phi=2\pi} \\
&= \left(\frac{1}{3}R^3\right)(4\pi) = \boxed{\frac{4}{3}\pi R^3}
\end{aligned}
$$

Hopefully this formula is already familiar to you, but this is another way to derive it!

Choose wisely

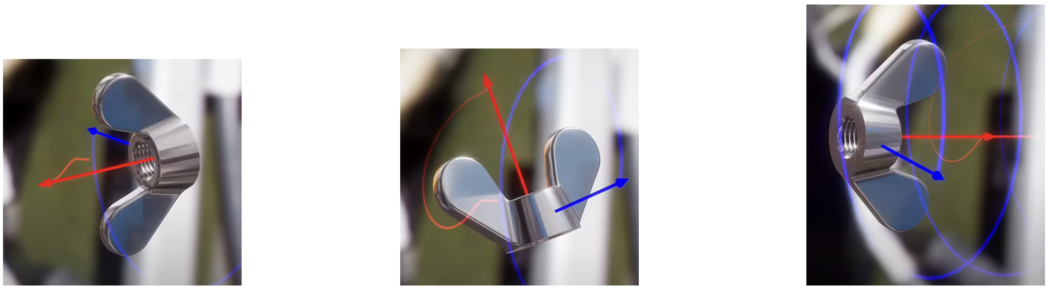
Remember to choose a co-ordinate system that fits your problem well.

3 The Tennis Racket Theorem and the end of the world?

This phenomenon goes by many names: the Tennis Racket Theorem, the Intermediate Axis Theorem and the Dzhanibekov Effect. We'll take a look at what it is, where it came from and why it happens. A student once asked Richard Feynman if there was an intuitive way to understand this phenomenon and the student was quickly met with a "No". Regardless, I'll try my best, drawing inspiration from Field medalist Terry Tao's explanation.

3.1 A Soviet state secret

In 1985, Soviet cosmonaut Vladimir Dzhanibekov was sent to the Salyut 7 space station as Russian scientists lost contact with it and it was beginning to drift out of orbit. His job was to repair it, and the supplies he had from Earth were secured using wing nuts. Unscrewing the wing nuts, he noticed a peculiar behaviour.



(a) Start of the motion (b) Flips all of a sudden? (c) Continues moving backwards

Figure 11: The effect as discovered by Dzhanibekov [13]

What he saw was that as he unscrewed the wing nut, it continued moving outwards from the screw, as expected due to the weightlessness in space. The unexpected moment was when the wing nut flipped all of a sudden while it was moving. As it continued to move, it would periodically flip 180 degrees - this is the Dzhanibekov effect. If you would like to see a video of this in action, a quick search will provide you with hundreds of results. It would be controversial to say that Dzhanibekov "discovered" this, as the effect was described almost 150 years earlier by the French mathematician and scientist Louis Poinsot, just not studied. When this was discovered, the Soviets kept it secret for around a decade. Their reason for secrecy was this: if a rotating body in space can flip all of a sudden, does that mean the Earth could potentially flip while travelling through space?

3.2 Why does it flip?

To explain this, we'll need some more knowledge about rotating bodies.

Principal moments of inertia and principal axes

For 3D rigid bodies, the moment of inertia can actually be represented as a tensor. A tensor can thought to be a matrix for now, since explaining the difference requires a considerable amount of linear algebra. Now, we can write the moment of inertia as:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

This is known as the inertia tensor. The terms along the diagonal (I_{xx}, I_{yy} and I_{zz}) are the moments of inertia about the x, y and z axes. The other terms are known as products of inertia; I will not go into too much detail about these but they quantify how mass is distributed around the axes. The inertia tensor depends on how you orient an object relative to the direction of its axis of rotation. It is possible to orient all 3D bodies in space such that their inertia tensor is of the form:

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

A matrix/tensor of this form is referred to as "diagonal". When the body is oriented in this way, the values of I_{xx}, I_{yy} and I_{zz} are known as the principal moments of inertia, and their respective axes are known as the principal axes. Let axis I_1 have moment of inertia I_{xx}, I_2 have moment of inertia I_{yy} and I_3 have moment of inertia I_{zz}. If $I_{xx} > I_{yy} > I_{zz}$, then I_1, I_2 and I_3 are known as the first, second and third principal axes respectively. This will be important. Some bodies that are symmetrical may have the same moment of inertia about two axes. It is important to mention now that our equations $\vec{L} = I\omega$ and $\tau = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$ assumed that we were in the principal frame i.e. the inertia tensor was diagonal.

The centrifugal force

We also need to gain an understanding of the centrifugal force, which is a fictitious or pseudo-force, which appears when we enter the frame of reference of a rotating body. This is demonstrated with a simple example: imagine a fairground ride, where you stand against a wall and the ride begins to spin. The floor disappears from below, yet you are stuck to the wall.

The ride operator sees you rotating and observes a centripetal force (the normal reaction force from the wall), causing you to move in a circular path - this is because he isn't rotating. You, on the other hand, feel like you are being pushed against the wall - this is because you are rotating - and the force you are feeling is the centrifugal force. The magnitude of the centrifugal force is equal to the magnitude of the centripetal force observed by the ride controller. Another fictional force, called the Coriolis force also appears in the rotating reference frame, but we shall not discuss this now.

Tying it all together

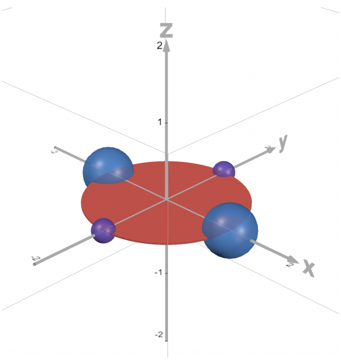


Figure 12: A disc with point masses m and M attached

Consider the body in Figure 12. It is a disc, with masses M in blue and masses m in purple attached to it. Since most of the mass is around the x-axis, rotating about the x-axis has the least moment of inertia. As a result, this is its first principal axis. Rotating about the z-axis would involve all 4 masses rotating, meaning rotating about the z-axis has the greatest moment of inertia, making it the third principal axis. This makes the y-axis the second principal axis, also known as: the intermediate axis. Sound familiar?

We'll show that rotating about the y-axis will eventually lead to the Dzhanibekov Effect. Imagine the body spinning about the y-axis. The small masses m stay in place, and thus experience no centripetal force. Now, the large masses M are rotating around the y-axis. Upon entering their rotating reference frame, we see that they are experiencing a centrifugal force equivalent to the centripetal force keeping them attached to the disc.

Now imagine that the disc is perturbed ever so slightly, such that the small masses m no longer lie on the y-axis exactly. This means that when rotating about the y-axis, the masses now experience a small centrifugal force, as they are no longer directly on the axis of rotation. You can imagine that the body has been ever so slightly rotated around the x-axis. As we set the disc into motion, the small masses m now experience a small centrifugal force causing them to accelerate even further away from the y-axis. As the masses move further away, they keep accelerating such that the body seems to rotate about the x-axis. Eventually, the body flips over the x-axis, and the centrifugal force switches direction, all while rotating about the y-axis.

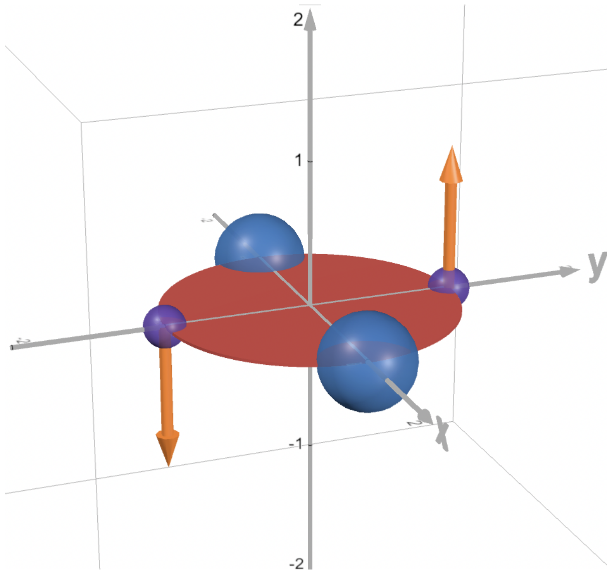


Figure 13: The centrifugal forces experienced by masses m after perturbation

Admittedly, this might not be easy to understand with just pictures and words. If you're interested, I recommend Veritasium's video "The Bizarre Behaviour of Rotating Bodies". You can in fact try this with your phone - there's one axis where its almost impossible to throw it and get a stable rotation. Which one?

3.3 We're safe for now

So why doesn't the Earth flip? The Earth is a spinning object in space right? It comes down to the fact that in space, angular momentum is conserved, but kinetic energy doesn't have to be. All systems will tend towards a state with the lowest amount of kinetic energy if they have any way of dissipating their energy, which the Earth does. Note that the rigid bodies we were discussing earlier had no means of losing their kinetic energy. Rotational kinetic energy is given by:

$$E_K = \frac{|\vec{L}|^2}{2I}$$

It follows that spinning about the axis with the maximum moment of inertia would lead to the lowest kinetic energy, since $\vec{L}$ is conserved. As a result, the Earth has settled into an axis of rotation which gives its the highest moment of inertia. It won't be flipping about its axis anytime soon.

4 Problems

Here are some interesting problems, either written by me, or collated from various sources. Each section roughly increases in difficulty. Solutions will be provided in the next issue.

Exploring rotational motion

1. Explain why it is easier to balance a cup on a finger upside down, with your finger inside the cup, as opposed to a cup the right way up, with your finger under the base.
2. For two point masses with mass m_1 and m_2 separated by a distance l rotating around their common centre of mass, find their:
   (a) moment of inertia
   (b) angular momentum, if one full rotation takes a time period T
3. Estimate the angular momentum of the Earth:
   (a) in its orbit around the Sun
   (b) on its axis
4. Prove that an unhinged, free body rotates about its centre of mass after a force has been applied to it.
5. Why might the cross-product definition for angular momentum break down in higher dimensions?

Co-ordinate systems

1. By integrating in a cylindrical co-ordinate system, find the volume of a frustum of a cone with base radius R, top radius r and height h.
2. Find the distance travelled by a particle in a time period τ, where the particle is travelling in a helical path described by the equations:

$$r = R$$
$$\theta = \omega t$$
$$z = vt$$

3. Calculate the moment of inertia for a cone with mass M, base radius R and height L around the axis perpendicular to base, passing through the apex.
4. Find the value of I where:

$$I = \int_{-\infty}^{\infty} e^{-x^2}$$

5. Which points in R^3 have the same co-ordinates in all 3 co-ordinate systems: Cartesian, cylindrical and spherical?

Credits

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